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Inequalities for Linear Combinations of Order Statistics from Restricted Families

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$$|K(x_i, y_j)|_{i,j=1}^{m} \geq 0.$$

For additional discussion see Karlin (1964).

We shall need the following total positivity result:

Lemma 3.5: Let f_{in} denote the density of the i^{th} order statistic in a sample of size n from F having density f. Then

(i) $f_{in}(x)$ is totally positive of order infinity (TP_∞) in $i=1,2,\ldots$ and $-\infty < x < \infty$,

(ii) $f_{in}(x)$ is RR_∞ in $n=1,2,\ldots$ and $-\infty < x < \infty$, and

(iii) $f_{n-i,n}(x)$ is TP_∞ in $n=1,2,\ldots$, and $-\infty < x < \infty$.

Proof: Note that

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(x)\bar{F}^{n-i}(x)f(x). \tag{3.5}$$

(i) Since $[F(x)/\bar{F}(x)]^{i-1}$ is TP_∞ in $i=1,2,\ldots,n$ and $-\infty < x < \infty$, then $f_{in}(x)$ is TP_∞ in i and x.

(ii) Since $[\bar{F}(x)]^n$ is RR_∞ in n and x, so is $f_{in}(x)$.

(iii) Since $f_{n-i,n}(x) = \frac{n!}{(n-i-1)!i!} F^{n-i-1}(x)\bar{F}^{i}(x)f(x)$ and $F^n(x)$ is TP_∞ in $n=1,2,\ldots$, and $-\infty < x < \infty$, the result follows.||

Using this total positivity property, we obtain

THEOREM 3.6: *Let $G^{-1}F$ be starshaped on the support of F and $F(0) = 0 = G(0)$. Then EX_{in}/EY_{in} is*

(i) decreasing in i,

(ii) increasing in n, and

(iii) $EX_{n-i,n}/EY_{n-i,n}$ is decreasing in n.

Manuscript prepared by Karen Harles

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ABSTRACT

Comparisons are obtained between a linear combination of order statistics from a distribution F and a corresponding linear combination from a distribution G where $G^{-1}F$ is (a) convex, and (b) starshaped. The results have applications in life testing where the underlying distribution has monotone failure rate or monotone failure rate on the average.

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1. INTRODUCTION

In this paper we present some results of theoretical interest concerning order statistics and their spacings from certain restricted families of positive random variables. Applications to life testing are discussed in a separate paper [Barlow and Proschan (in process)].

For a specified continuous distribution G for which $G(0) = 0$, we consider the family $\mathcal{G}$ of distributions such that for F in $\mathcal{G}$ and $F(0) = 0$, $G^{-1}F$ is starshaped or convex on the support of F. Distributions related in this way by convexity have been studied by Van Zwet (1964). It is known that $F(0) = 0$, $G(0) = 0$, and $G^{-1}F$ convex imply $G^{-1}F$ starshaped. [Bruckner and Ostrow (1962)].

If G is the exponential distribution, then $G^{-1}F$ convex where finite is equivalent to F having an increasing failure rate (i.e., F is IFR). $G^{-1}F$ starshaped is equivalent to F having an increasing failure rate average (i.e., F is IFRA) [Birnbaum, Esary, and Marshall (1965)]. $G^{-1}F$ concave on $[0,\infty)$ is equivalent to F having decreasing failure rate (i.e., F is DFR).

If G is the uniform distribution, then $G^{-1}F$ convex on the support of F is equivalent to F having an increasing density. If F (G) denotes the gamma distribution with shape parameter α (β) with $\alpha > \beta$, then $G^{-1}F$ is convex on $[0,\infty)$ [Van Zwet (1964)]. The Weibull family is similarly ordered, as may be readily verified.

Comparisons for linear combinations of expected values of order statistics from F and G are obtained when $G^{-1}F$ is starshaped. In addition, stochastic comparisons for linear combinations of order statistics are obtained when $G^{-1}F$ is convex as well as when $G^{-1}F$ is starshaped.

Specializing to the case where G is the exponential distribution and F is IFR or IFRA, stochastic comparisons are made for the "total time on test", which is of interest in life testing. Bounds on the expected values of order statistics are also obtained for this case.

Finally, we investigate the preservation of certain class properties under the operation of taking order statistics.

2. PRELIMINARIES

Throughout this paper we adopt the following notation and assumptions. Let X (Y) have distribution F (G). We assume that $F(0) = 0 = G(0)$, and that F and G are continuous. We also assume that the support of F is an interval, possibly infinite, and that G is strictly increasing on its support. We use $\bar{F}$ for $1-F$ and $\bar{G}$ for $1-G$.

We consider functions ϕ defined on $[0,b]$, $0 < b \leq \infty$. ϕ is starshaped on $[0,b]$ if $\phi(\alpha x) \leq \alpha\phi(x)$ for $0 \leq \alpha \leq 1$, $0 \leq x \leq b$ (or equivalently, if $\phi(x)/x$ is increasing for x in $[0,b]$); and ϕ is convex on $[0,b]$ if $\phi[\alpha x + (1-\alpha)y] \leq \alpha\phi(x) + (1-\alpha)\phi(y)$ for $0 \leq \alpha \leq 1$, $0 \leq x, y \leq b$. Then on $[0,b]$, convex ϕ such that $\phi(0) \leq 0$ are starshaped.

The following properties of IFR (DFR) distributions will be needed [cf. Barlow and Proschan (1965), Chapter II]. If F is IFR (DFR) and

$$G(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases},$$

then

(i) $G^{-1}F$ is convex (concave) where finite (on $[0,\infty)$);

(ii) $F_u(x) = [F(x+u) - F(u)]/\bar{F}(u)$ is increasing (decreasing) in $u \geq 0$ for all $x \geq 0$ whenever the denominator is nonzero;

(iii) $G^{-1}F$ $(F^{-1}G)$ is starshaped where defined (on $[0,\infty)$);

(iv) $[\bar{F}(x)]^{\frac{1}{x}}$ is decreasing (increasing) in $x \geq 0$.

Let $X_{1n} \leq \cdots \leq X_{nn}$ $(Y_{1n} \leq \cdots \leq Y_{nn})$ denote an ordered sample of size n from F (G); define $X_{on} \equiv 0$ $(Y_{on} \equiv 0)$. We drop the second subscript when there is no danger of confusion. We use the term increasing (decreasing) for nondecreasing (nonincreasing). We use the notation $\overset{st}{\geq}$ $\left(\overset{st}{\leq}\right)$ for "stochastically greater than" ("stochastically less than") and $\overset{st}{=}$ for "stochastically equivalent to".

3. INEQUALITIES IN THE CASE OF STARSHAPEDNESS

In this section we consider pairs of distributions F and G such that $G^{-1}F$ is starshaped on the support of F. We shall obtain a stochastic comparison between linear combinations of order statistics $X_{1n} \leq \cdots \leq X_{nn}$ from F and $Y_{1n} \leq \cdots \leq Y_{nn}$ from G. To do this we first present some basic inequalities for starshaped functions. For further discussion and extension of Lemmas 3.1 and 3.3, see Barlow, Marshall,

and Proschan (in process). We shall find it convenient to define $\bar{A}_i = \sum_{j=i}^{n} a_j$.

Lemma 3.1:
$$\phi\left(\sum_{i=1}^{n} a_i x_i\right) \le \sum_{i=1}^{n} a_i \phi(x_i) \qquad (3.1)$$

for all starshaped ϕ on $[0,b]$ and all $0 \le x_1 \le \cdots \le x_n \le b$ if and only if there exists k $(1 \le k \le n)$ such that $0 \le \bar{A}_1 \le \cdots \le \bar{A}_k \le 1$ and $\bar{A}_{k+1} = \cdots = \bar{A}_n = 0$.

Proof:

Sufficiency. Assume $0 \le \bar{A}_1 \le \cdots \le \bar{A}_k \le 1$ and $\bar{A}_{k+1} = \cdots = \bar{A}_n = 0$. Then $a_i \le 0$ for $i=1,2,\ldots,k-1$, $0 \le a_k \le 1$, $a_i = 0$ for $i=k+1,\ldots,n$. Using the identity $\sum_1^n a_i x_i \equiv \sum_1^n \bar{A}_i(x_i - x_{i-1})$, we conclude that $0 \le \sum_1^n a_i x_i \le x_k$. Thus

$$\phi(x_k)/x_k \ge \phi(x_i)/x_i \qquad \text{for } i=1,\ldots,k-1, \text{ and}$$

$$\phi(x_k)/x_k \ge \phi(\Sigma a_i x_i)/\Sigma a_i x_i.$$

Hence
$$\left\{\sum_1^{k-1} (-a_i)x_i + \sum_1^n a_i x_i\right\} \frac{\phi(x_k)}{x_k} \ge \sum_1^{k-1} (-a_i)\phi(x_i) + \phi(\Sigma a_i x_i),$$

or
$$a_k \phi(x_k) \ge \sum_1^{k-1} (-a_i)\phi(x_i) + \phi(\Sigma a_i x_i).$$

Necessity. Let $\phi(x) = x^2$, $0 = x_1 = \cdots = x_{i-1}$, and $x_i = \cdots = x_n = 1$. Then (3.1) implies

$$\left(\sum_{j=i}^{n} a_j\right)^2 \le \sum_{j=i}^{n} a_j, \quad \text{so that } 0 \le \bar{A}_i \le 1 \text{ for } i=1,2,\ldots,n.$$

Next we shall show that $\bar{A}_j > 0$ implies $\bar{A}_{j-1} \leq \bar{A}_j$. To see this, let $0 = x_1 = \cdots = x_{j-2} < x_{j-1} < x_j = x_{j+1} = \cdots = x_n$. Then $\sum_1^n a_i x_i = \sum_1^n \bar{A}_i (x_i - x_{i-1}) = \bar{A}_{j-1} x_{j-1} + \bar{A}_j (x_j - x_{j-1})$. Let $x_{j-1} < z < x_j$ and x_j be so large that $\sum_1^n a_i x_i > z$. Let

$$\phi_z(x) = \begin{cases} 0 & x < z \\ x & x \geq z, \end{cases}$$

a starshaped function. From (3.1),

$$\phi_z\left(\sum_1^n a_i x_i\right) = \bar{A}_{j-1} x_{j-1} + \bar{A}_j (x_j - x_{j-1}) \leq \bar{A}_j x_j .$$

This implies $\bar{A}_{j-1} - \bar{A}_j \leq 0$.

Finally let k denote the largest subscript i, if it exists, such that $\bar{A}_i > 0$. Assume that $\bar{A}_{j+1} = 0$ for $j < k-1$. We shall show that this implies $\bar{A}_i = 0$ for $i \leq j$. Let $x_j < z \leq x_{j+1}$ and x_k be so large that $\sum_1^n a_i x_i = \sum_1^k \bar{A}_i (x_i - x_{i-1}) > z$. Then

$$\phi_z\left(\sum_1^n a_i x_i\right) = \sum_1^n a_i x_i \leq \sum_{j+1}^n a_i x_i$$

which implies $\sum_1^j a_i x_i = \sum_1^j \bar{A}_i (x_i - x_{i-1}) + \bar{A}_{j+1} x_j \leq 0$. This in turn implies $\bar{A}_i = 0$ for $i = 1, 2, \ldots, j$ since $\bar{A}_{j+1} = 0$ and $0 \leq \bar{A}_i \leq 1$. ||

THEOREM 3.2: *Let $G^{-1}F$ be starshaped on the support of F, $F(0) = 0 = G(0)$. If there exists k $(1 \leq k \leq n)$ such that*

$$0 \leq \bar{A}_1 \leq \bar{A}_2 \leq \cdots \leq \bar{A}_k \leq 1 \quad \text{and} \quad \bar{A}_{k+1} = \cdots = \bar{A}_n = 0,$$

then
$$F\left(\sum_1^n a_i X_i\right) \overset{st}{\leq} G\left(\sum_1^n a_i Y_i\right). \tag{3.2}$$

Proof: By Lemma 3.1,

$$G^{-1}F\left(\sum_1^n a_i X_i\right) \leq \sum_1^n a_i G^{-1}F(X_i) \overset{st}{=} \sum_1^n a_i Y_i .$$

The stochastic equivalence follows from Lemma 1, p. 73, Lehman (1959). Hence (3.2) holds.||

Theorem 3.2 can be used to obtain conservative lower tolerance limits [Barlow, Proschan (1966)].

To obtain a reverse inequality to that of (3.2) we need the following lemma:

Lemma 3.3:

$$\phi\left(\sum_1^n a_i x_i\right) \geq \sum_1^n a_i \phi(x_i) \tag{3.3}$$

for all $0 \leq x_1 \leq \cdots \leq x_n$ and for all starshaped ϕ on $(-\infty,\infty)$ if and only if there exists k $(1 \leq k \leq n)$ such that

$$\bar{A}_1 \geq \bar{A}_2 \geq \cdots \geq \bar{A}_k \geq 1; \quad \bar{A}_{k+1} = \cdots = \bar{A}_n = 0,$$

or equivalently,

$a_i \geq 0$ for $1 \leq i \leq k$; $a_k \geq 1$ and $a_i = 0$ for $i=k+1,\ldots,n$.

Proof:

Sufficiency. We may assume $x_1 > 0$. Hence

$$\phi\left(\sum_1^k a_i x_i\right) / \sum_1^k a_i x_i \geq \phi(x_i)/x_i$$

for $i=1,2,\ldots,k$, since ϕ is starshaped and $\sum_1^k a_i x_i \geq x_k$. It follows that $\sum_1^k a_i x_i \phi\left(\sum_1^k a_i x_i\right) / \sum_1^k a_i x_i \geq \sum_1^k a_i \phi(x_i)$, yielding (3.3).

Necessity. From the proof of necessity in Lemma 3.1 we see that for each i, either $\bar{A}_i \leq 0$ or $\bar{A}_i \geq 1$. First we claim $\bar{A}_i \leq 0$ implies $\bar{A}_{i+1} \leq 0$, and hence $\bar{A}_{i+1} \geq 1$ implies $\bar{A}_i \geq 1$. To see this, let $0 = x_1 = \cdots = x_{i-1} < x_i < z < x_{i+1} = \cdots = x_n$. Choose $(x_{i+1} - x_i)$ sufficiently small so that

$$\sum_1^n a_i x_i = \bar{A}_i x_i + \bar{A}_{i+1}(x_{i+1} - x_i) \leq x_i .$$

We can do this since by assumption $\bar{A}_i \leq 0$. Hence by (3.3), $\phi_z\left(\sum_1^n a_i x_i\right) = 0 \geq \bar{A}_{i+1} x_{+1}$, which implies $\bar{A}_{i+1} \leq 0$. Thus we have shown that for some k $(0 \leq k \leq n)$

$$\bar{A}_1 \geq 1,\ldots,\bar{A}_k \geq 1; \quad \bar{A}_{k+1} \leq 0,\ldots,\bar{A}_n \leq 0.$$

We claim that we cannot have $\bar{A}_i \geq 1$ and $\bar{A}_j < 0$ for $j > i$. Suppose this were the case. Choose

$$0 = x_1 = \cdots = x_{i-1} < z < x_i = x_{i+1} = \cdots = x_{j-1} < x_j = \cdots = x_n,$$

and $(x_j - x_i)$ in such a manner that

$$0 < \sum_1^n \bar{A}_i (x_i - x_{i-1}) = \bar{A}_i x_i + \bar{A}_j (x_j - x_i) < z.$$

Then by (3.3)

$$\phi_z \left[\sum_1^n a_i x_i \right] = 0 \geq \bar{A}_i x_i + \bar{A}_j (x_j - x_i) > 0,$$

which is a contradiction. It follows that $\bar{A}_{k+1} = \cdots = \bar{A}_n = 0$.

Next we shall show that $\bar{A}_{i+1} \geq 1$ implies $\bar{A}_i \geq \bar{A}_{i+1}$. To see this, let $0 = x_1 = \cdots = x_{i-1} < x_i < z < x_{i+1} = \cdots = x_n$. Then

$$\sum_1^n a_i x_i = \bar{A}_i x_i + \bar{A}_{i+1} (x_{i+1} - x_i) \geq x_{i+1},$$

which implies by (3.3) that

$$\phi_z \left(\sum_1^n a_i x_i \right) = \bar{A}_i x_i + \bar{A}_{i+1} (x_{i+1} - x_i) \geq \bar{A}_{i+1} x_{i+1}.$$

This implies $\bar{A}_i \geq \bar{A}_{i+1}$. ||

From Lemma 3.3 we obtain

THEOREM 3.4: *Let $G^{-1}F$ be starshaped on the support of F and $F(0) = 0 = G(0)$. Let $a_i \geq 0$ for $i=1,2,\ldots,n$, and $a_n \geq 1$. Then*

$$F\left(\sum_1^n a_i X_i\right) \overset{st}{\geq} G\left(\sum_1^n a_i Y_i\right). \tag{3.4}$$

Proof: By assumption, the support of F is an interval, say $[0,b]$. If $\sum_1^n a_i X_i > b$, the result is obvious. Hence we may suppose $\sum_1^n a_i X_i \leq b$. Apply Lemma 3.3 to obtain

$$G^{-1}F\left(\sum_1^n a_i X_i\right) \geq \sum_1^n a_i G^{-1}F(X_i) \overset{st}{=} \sum_1^n a_i Y_i.$$

The stochastic equivalence follows from Lemma 1, p. 73, Lehman (1959). Hence (3.4) holds.||

Theorem 3.4 can be used to obtain conservative upper tolerance limits [Barlow, Proschan (1966)].

Next we obtain results concerning expected values. We shall need the concept of total positivity. A function $K(x,y)$ of two real variables $x \in X$, $y \in Y$, where X and Y are ordered sets, is said to be *totally positive of order* r (TP_r) if for all $1 \leq m \leq r$, $x_1 \leq x_2 \leq \cdots \leq x_m$, and $y_1 \leq y_2 \leq \cdots \leq y_m$, where each $x_i \in X$, $y_i \in Y$, we have the determinantal inequalities

$$\left| K(x_i, y_j) \right|_{i,j=1}^{m} \geq 0.$$

$K(x,y)$ is said to be *reverse regular of order* r (RR_r) if for every $1 \leq m \leq r$, $x_1 \geq x_2 \geq \cdots \geq x_m$, $y_1 \leq y_2 \leq \cdots \leq y_m$, where each $x_i \in X$, $y_j \in Y$,

$$|K(x_i, y_j|_{i,j=1}^{m} \geq 0.$$

For additional discussion see Karlin (1964).

We shall need the following total positivity result:

Lemma 3.5: Let f_{in} denote the density of the i^{th} order statistic in a sample of size n from F having density f. Then

(i) $f_{in}(x)$ is totally positive of order infinity (TP_∞) in $i=1,2,\ldots$ and $-\infty < x < \infty$,

(ii) $f_{in}(x)$ is RR_∞ in $n=1,2,\ldots$ and $-\infty < x < \infty$, and

(iii) $f_{n-i,n}(x)$ is TP_∞ in $n=1,2,\ldots$, and $-\infty < x < \infty$.

Proof: Note that

$$f_{in}(x) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(x)\bar{F}^{n-i}(x)f(x). \tag{3.5}$$

(i) Since $[F(x)/\bar{F}(x)]^{i-1}$ is TP_∞ in $i=1,2,\ldots,n$ and $-\infty < x < \infty$, then $f_{in}(x)$ is TP_∞ in i and x.

(ii) Since $[\bar{F}(x)]^n$ is RR_∞ in n and x, so is $f_{in}(x)$.

(iii) Since $f_{n-i,n}(x) = \frac{n!}{(n-i-1)!i!} F^{n-i-1}(x)\bar{F}^{i}(x)f(x)$ and $F^n(x)$ is TP_∞ in $n=1,2,\ldots$, and $-\infty < x < \infty$, the result follows.||

Using this total positivity property, we obtain

THEOREM 3.6: *Let $G^{-1}F$ be starshaped on the support of F and $F(0) = 0 = G(0)$. Then EX_{in}/EY_{in} is*

(i) decreasing in i,

(ii) increasing in n, and

(iii) $EX_{n-i,n}/EY_{n-i,n}$ is decreasing in n.

Proof:

(i) Let $\phi(x) = G^{-1}F(x)$. Since ϕ is starshaped, then for arbitrary $c \geq 0$, $x - c\phi(x)$ changes sign at most once, and from positive to negative values if at all. Define

$$h(i) = \int_0^\infty [x - c\phi(x)]f_{in}(x)dx = EX_i - cEY_i.$$

Since $f_{in}(x)$ is TP_∞ in i and x, then $h(i)$ changes sign at most once, and from positive to negative values if at all, by the variation diminishing property of totally positive functions [Karlin (1964), p. 34]. Hence $[EX_i/EY_i] - c$ changes sign at most once, and from positive to negative values if at all. Since $c \geq 0$ is arbitrary, the ratio EX_i/EY_i is decreasing in i.

(ii) Since $f_{in}(x)$ is RR_∞ in n and x, by using a similar argument we may show that EX_{in}/EY_{in} is increasing in n.

(iii) Since $f_{n-i,n}(x)$ is TP_∞ in n and x, by using a similar argument we may obtain the desired conclusion.||

Choosing G to be the uniform distribution we see from Theorem 3.6 that $(n + 1)EX_{in}/i$ is decreasing in i and increasing in n, where $X_{1n} \leq \cdots \leq X_{nn}$ are order statistics from F, a distribution with increasing density. Choosing G to be the exponential distribution, we see that $EX_{in}/\sum_{j=1}^{i} \frac{1}{n-j+1}$ is decreasing in i and increasing in n, where $X_{1n} \leq \cdots \leq X_{nn}$ are order statistics from F, an IFRA distribution.

By using Theorem 3.6, bounds on EX_{in} can be obtained as follows.

Note that

$$EX_{ii}/EY_{ii} \le EX_{in}/EY_{in} \le EX_{1,n-i+1}/EY_{1,n-i+1}, \tag{3.6}$$

the first inequality from Theorem 3.6 (ii), the second from Theorem 3.6 (iii). Now suppose that

$$\int_0^\infty xdF(x) = \int_0^\infty xdG(x) = \theta.$$

This implies

$$\theta EY_{in}/EY_{ii} \le EX_{in} \le \theta EY_{in}/EY_{1,n-i+1}. \tag{3.7}$$

To obtain an application of Theorem 3.6 (iii) we choose G uniform on $[0,1]$ and F such that f is increasing on the support of F. Then we immediately have

$$\frac{(n+1)EX_{n-i,n}}{(n-i)}$$

decreasing in $n \ge i+1$ $(i=0,1,\ldots)$. This is a strengthening of the monotonicity result of Corollary 4 of Marshall and Proschan (1965) which implies as a special case that EX_{nn}/n is decreasing in n whenever the underlying distribution F satisfies $F(0^-) = 0$.

We will need the following lemma:

Lemma 3.7: If $\frac{\beta_i}{\alpha_i}$ is increasing in i $(1 \le i \le n)$, and $0 \le \alpha_1 \le \cdots \le \alpha_n$, $\beta_i \ge 0$, then

$$\text{(i)} \quad \sum_1^r \beta_i / \sum_1^r \alpha_i \quad \text{and} \quad \frac{\sum_1^r (n-i+1)(\beta_i - \beta_{i-1})}{\sum_1^r (n-i+1)(\alpha_i - \alpha_{i-1})}$$

are increasing in r $(1 \le r \le n)$; in particular,

$$\text{(ii)} \quad \frac{\sum_1^r \beta_i}{\sum_1^n \beta_i} \le \frac{\sum_1^r \alpha_i}{\sum_1^n \alpha_i} \quad \text{and} \quad \frac{\sum_1^r (n-i+1)(\beta_i - \beta_{i-1})}{\sum_1^n \beta_i} \le \frac{\sum_1^r (n-i+1)(\alpha_i - \alpha_{i-1})}{\sum_1^n \alpha_i};$$

$$\text{(iii)} \quad \frac{\sum_1^n a_i \beta_i}{\sum_1^n \beta_i} \le \frac{\sum_1^n a_i \alpha_i}{\sum_1^n \alpha_i} \quad \text{and} \quad \frac{\sum_1^n a_i (n-i+1)(\beta_i - \beta_{i-1})}{\sum_1^n \beta_i} \le \frac{\sum_1^n a_i (n-i+1)(\alpha_i - \alpha_{i-1})}{\sum_1^n \alpha_i}$$

if $a_1 \ge a_2 \ge \cdots \ge a_n$.

Proof: Define $\psi(0) = 0$, $\psi(\alpha_1 + \cdots + \alpha_i) = \beta_1 + \cdots + \beta_i$ $(1 \le i \le n)$. Define $\psi(x)$ elsewhere on $[0, \alpha_1 + \cdots + \alpha_n]$ by linear interpolation between successive points defined above. Note that

$$\frac{\psi(\alpha_1 + \cdots + \alpha_i) - \psi(\alpha_1 + \cdots + \alpha_{i-1})}{(\alpha_1 + \cdots + \alpha_i) - (\alpha_1 + \cdots + \alpha_{i-1})} = \frac{\beta_i}{\alpha_i}$$

is increasing in i, so that ψ is a convex function on $[0, \alpha_1 + \cdots + \alpha_n]$. Since $\psi(0) = 0$, ψ is also starshaped, i.e. $\frac{\psi(x)}{x}$ is increasing in x.

Hence

$$\frac{\psi\left[\sum_1^r \alpha_i\right]}{\sum_1^r \alpha_i} = \frac{\sum_1^r b_j}{\sum_1^r \alpha_i}$$

is increasing in r.

To show $\dfrac{\sum_1^r \beta_i + (n-r)\beta_r}{\sum_1^r \alpha_i + (n-r)\alpha_r}$ is increasing in r,

define $f_1(x) = \dfrac{\beta_r}{\alpha_r} x + b_1$

$$f_2(x) = \frac{\beta_{r+1}}{\alpha_{r+1}} x + b_2$$

where b_1 and b_2 are chosen to satisfy $f_1(\alpha_1 +\cdots+ \alpha_r) = f_2(\alpha_1 +\cdots+ \alpha_r) = \beta_1 +\cdots+ \beta_r$, so that

$$f_1[\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_r] = \beta_1 +\cdots+ \beta_r + (n-r)\beta_r$$

and

$$f_2[\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_{r+1}] = \beta_1 +\cdots+ \beta_r + (n-r)\beta_{r+1}.$$

Since $\dfrac{\beta_{r+1}}{\alpha_{r+1}} \geq \dfrac{\beta_r}{\alpha_r}$, it follows that $f_1(x) \leq f_2(x)$ for $x \geq \alpha_1 +\cdots+ \alpha_r$.

Hence

$$\frac{f_1[\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_r]}{\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_r} \leq \frac{f_2[\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_r]}{\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_r}$$

$$\leq \frac{f_2[\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_{r+1}]}{\alpha_1 +\cdots+ \alpha_r + (n-r)\alpha_{r+1}} .$$

The last inequality follows since f_2 is tangent to a starshaped function and therefore is starshaped, and since

$$\alpha_1 + \cdots + \alpha_r + (n-r)\alpha_{r+1} \geq \alpha_1 + \cdots + \alpha_r + (n-r)\alpha_r.$$

This proves (i). (ii) is an immediate consequence of (i).

To see (iii), let $\sum_1^n a_i \left[\frac{\beta_i}{\Sigma \beta_i} - \frac{\alpha_i}{\Sigma \alpha_i} \right] = \sum_1^n a_i d_i$, and note that

$$\sum_1^n a_i d_i = (a_1 - a_2)d_1 + (a_2 - a_3)(d_1 + d_2) +$$

$$+ a_n(d_1 + \cdots + d_n).$$

Since $a_{i-1} - a_i \geq 0$, $i=1,2,\ldots,n-1$, $d_1 + \cdots + d_i \leq 0$ $(i=1,2,\ldots,n-1)$, and $d_1 + \cdots + d_n = 0$, we conclude that

$$\sum_1^n a_i d_i \leq 0.$$

The second inequality in (iii) is proved similarly.||

To prove the next result, we need to introduce the following concepts:

<u>Definition</u>: A sequence $a = (a_1, \ldots, a_n)$ is said to *majorize* a sequence $b = (b_1, \ldots, b_n)$ (written $a \succ b$) if $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$, and $\sum_1^r a_i \geq \sum_1^r b_i$ for $r=1,\ldots,n-1$, while $\sum_1^n a_i = \sum_1^n b_i$.

This definition differs slightly from that of Hardy, Littlewood, Pólya (1959), p. 45, but corresponds to the usage of Beckenbach and Bellman (1961), p. 30.

<u>Definition</u>: If a differentiable function $H(z_1,\ldots,z_n)$ satisfies

$$(z_i - z_j)\left(\frac{\partial H}{\partial z_i} - \frac{\partial H}{\partial z_j}\right) \geq 0$$

for all $\underline{z}$, i, j, then H is said to satisfy the *Schur condition.*

We shall use the following theorem (see Ostrowski (1952)):

THEOREM 3.8: *(Schur, Ostrowski) Assume* H *is defined for* $z_1 \geq \cdots \geq z_n$ *and has partial derivatives. Then* $H(\underline{z}) \geq H(\underline{z}')$ *for all* $\underline{z} \succ \underline{z}'$ *if and only if* $H(\underline{z})$ *satisfies the Schur condition.*

THEOREM 3.9: *Let* $G^{-1}F$ *be starshaped on the support of* F, $F(0) = G(0) = 0$, *and* $\int_0^\infty x\,dF(x) = \int_0^\infty x\,dG(x)$. *Then*

$$(i)\quad \sum_1^r EY_i \Big/ \sum_1^r EX_i \quad \text{and} \quad \sum_1^r (n-i+1)E(Y_i-Y_{i-1}) \Big/ \sum_1^r (n-i+1)E(X_i-X_{i-1})$$

are increasing in r $(1 \leq r \leq n)$;

$$(ii)\quad (EY_n, EY_{n-1}, \ldots, EY_1) \succ (EX_n, EX_{n-1}, \ldots, EX_1)$$

and $\sum_1^r (n-i+1)E(X_i-X_{i-1}) \geq \sum_1^r (n-i+1)E(Y_i-Y_{i-1})$ *for* $1 \leq r \leq n$;

$$(iii)\quad H(EY_n, EY_{n-1}, \ldots, EY_1) \geq H(EX_n, EX_{n-1}, \ldots, EX_1)$$

if H *is a Schur function;*

(iv) $$\sum_1^n a_i(n-i+1)E(X_i-X_{i-1}) \geq \sum_1^n a_i(n-i+1)E(Y_i-Y_{i-1})$$

if $a_1 \geq a_2 \geq \cdots \geq a_n$.

Proof: Since $\frac{EY_i}{EX_i}$ is increasing in i by Theorem 3.6, (i) follows from Lemma 3.7 (i). Since $\int_0^\infty x dF(x) = \int_0^\infty x dG(x)$, (ii) follows from Lemma 3.7 (ii). (iii) follows from (ii) and Theorem 3.8. (iv) follows from Lemma 3.7 (iii).||

The following result presented in Hardy, Littlewood, and Pólya (1959), p. 89, is used to obtain Corollary 3.11.

THEOREM 3.10: *If* ϕ *is convex on the interval* I *and* $\underline{x} \succ \underline{y}$, *where* $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ *belong to* I, *then* $\sum_1^n \phi(x_i) \geq \sum_1^n \phi(y_i)$.

Corollary 3.11: Let $G^{-1}F$ be starshaped on the support of F, $F(0) = 0 = G(0)$, $\int_0^\infty x dF(x) = \int_0^\infty x dG(x)$, and ψ be convex. Then $\sum_1^n \psi(EY_i) \geq \sum_1^n \psi(EX_i)$.

Proof: The result follows immediately from Theorem 3.9 (ii) and Theorem 3.10.||

The following theorem is obtained in Marshall, Olkin, and Proschan (1966).

THEOREM 3.12: *If* $G^{-1}F$ *is starshaped on the support of* F, *then*

(i) $$\left(\frac{X_n}{\Sigma X_i},\ldots,\frac{X_1}{\Sigma X_i}\right) \overset{st}{\prec} \left(\frac{Y_n}{\Sigma Y_i},\ldots,\frac{Y_1}{\Sigma Y_i}\right)$$

(ii) $$H\left(\frac{X_n}{\Sigma X_i},\ldots,\frac{X_1}{\Sigma X_i}\right) \overset{st}{\leq} H\left(\frac{Y_n}{\Sigma Y_i},\ldots,\frac{Y_1}{\Sigma Y_i}\right)$$

if H is a Schur function;

(iii) $$\sum_1^r \frac{(n-i+1)(X_i-X_{i-1})}{\bar{X}} \overset{st}{\geq} \sum_1^r \frac{(n-i+1)(Y_i-Y_{i-1})}{\bar{Y}}$$

(iv) $$\left(\frac{1}{n}\sum_1^n X_i^2 - \bar{X}^2\right)/\bar{X}^2 \overset{st}{\leq} \left(\frac{1}{n}\sum_1^n Y_i^2 - \bar{Y}^2\right)/\bar{Y}^2.$$

(v) If, in addition, $a_1 \geq \cdots \geq a_n$, then

$$\sum_1^n a_i X_i/\bar{X} \overset{st}{\geq} \sum_1^n a_i Y_i/\bar{Y}$$

and $$\sum_1^n \frac{a_i(n-i+1)(X_i-X_{i-1})}{\bar{X}} \overset{st}{\geq} \sum_1^n \frac{a_i(n-i+1)(Y_i-Y_{i-1})}{\bar{Y}}$$

where $\bar{X} = \sum_1^n X_i/n$ and $\bar{Y} = \sum_1^n Y_i/n$.

<u>Proof</u>: By definition $\frac{G^{-1}F(X_i)}{X_i} = \frac{Y_i'}{X_i}$ is increasing in i. Hence (i) follows from Lemma 3.7 (i) and Lehman (1959), p. 73. (ii) follows from (i) and Theorem 3.8. (iii) is a consequence of Lemma 3.7 (ii). (iv) follows from (ii) where

$$H(z_1,\ldots,z_n) = n^{-1}\sum_1^n z_i^2 - 1.$$

(v) follows from Lemma 3.7 (iii).||

4. INEQUALITIES IN THE CASE OF CONVEXITY

In this section we consider pairs of distributions F and G such that $G^{-1}F$ is convex on the support of F and $F(0) = 0 = G(0)$. This is a strengthening of the starshapedness hypothesis of Section 3. Our first result has applications to conservative lower tolerance limits [Barlow and Proschan (1966)]. We shall need the following inequality which is of independent interest. See Barlow, Marshall, and Proschan (in process) for further discussion and extension of Lemmas 4.1 and 4.3.

Lemma 4.1:

$$\text{(i)}\quad \phi\left(\sum_1^n a_i x_i\right) - \phi(0) \leq \sum_1^n a_i[\phi(x_i) - \phi(0)] \tag{4.1}$$

for all $0 \leq x_1 \leq \cdots \leq x_n \leq b$ and for all convex ϕ on $[0,b]$ if and only if $0 \leq \bar{A}_i \equiv \sum_{j=i}^n a_j \leq 1$ for $i=1,2,\ldots,n$.

(ii) If (4.1) holds for all $\underline{a}$ satisfying $0 \leq \bar{A}_i \leq 1$, $i=1,\ldots,n$, then ϕ is convex on $[0,b]$.

<u>Proof</u>:

(i) *Sufficiency*. First assume $\phi(0) = 0$ and $0 \le \bar{A}_i \le 1$ for $i=1,\ldots,n$. Then

$$\phi\left(\sum_1^n a_i x_i\right) = \phi\left(\sum_1^n \bar{A}_i (x_i - x_{i-1})\right) = \sum_{j=1}^n \left|\phi\left(\sum_{i=1}^j \bar{A}_i (x_i - x_{i-1})\right) - \phi\left(\sum_{i=1}^{j-1} \bar{A}_i (x_i - x_{i-1})\right)\right|$$

(where $\sum_{i=1}^{0} \equiv 0$). Since the $\bar{A}_j$ are ≤ 1, the last expression is

$$\le \sum_{j=1}^n \bar{A}_j \left|\phi\left(\sum_{i=1}^{j-1} \bar{A}_i (x_i - x_{i-1}) + (x_j - x_{j-1})\right) - \phi\left(\sum_{i=1}^{j-1} \bar{A}_i (x_i - x_{i-1})\right)\right|$$

$$\le \sum_{j=1}^n \bar{A}_j \left|\phi\left(\sum_{i=1}^{j-1} (x_i - x_{i-1}) + (x_j - x_{j-1})\right) - \phi\left(\sum_{i=1}^{j-1} (x_i - x_{i-1})\right)\right|$$

$$= \sum_{j=1}^n \bar{A}_j [\phi(x_j) - \phi(x_{j-1})] = \sum_{j=1}^n a_j \phi(x_j).$$

Note that if we let $\psi(x) = \phi(x) - \phi(0)$ where ϕ is convex, then ψ is convex and $\psi(0) = 0$. Hence (4.1) holds for all convex ϕ on $[0,b]$.

(i) *Necessity*. Next assume (4.1) holds. Choose $\phi(x) = x^2$ and $0 = x_1 = \cdots = x_{j-1}$; $x_j = \cdots = x_n = 1$. Then (4.1) implies $\left(\sum_j^n a_i\right)^2 \le \sum_j^n a_i$, so that $0 \le \bar{A}_j \le 1$.

(ii) Now suppose (4.1) holds for all $\underline{a}$ satisfying $0 \le \bar{A}_i \le 1$ for $i=1,2,\ldots,n$. Then $\phi(x) - \phi(0)$ is convex on $[0,b]$ directly from the definition of convexity. Hence ϕ is convex on $[0,b]$. ||

We may now prove

THEOREM 4.2: *Let $G^{-1}F$ be convex on the support of F, $F(0) = 0 = G(0)$, and $0 \leq \bar{A}_i \leq 1$ for $i=1,\ldots,n$. Then*

$$F\left(\sum_1^n a_i X_i\right) \overset{st}{\leq} G\left(\sum_1^n a_i Y_i\right), \tag{4.2}$$

or equivalently,

$$F\left[\sum_1^n \bar{A}_i (X_i - X_{i-1})\right] \overset{st}{\leq} G\left[\sum_1^n \bar{A}_i (Y_i - Y_{i-1})\right] \tag{4.3}$$

<u>Proof</u>: Using Theorem 4.1, we have

$$G^{-1}F\left(\sum_1^n a_i x_i\right) \leq \sum_1^n a_i G^{-1}F(X_i) \overset{st}{=} \sum_1^n a_i Y_i .$$

The stochastic equivalence follows from Lemma 1, p. 73, Lehman (1959). Thus (4.2) follows.

The equivalence of (4.3) and (4.2) follows from the identity $\sum_1^n a_i x_i = \sum_1^n \bar{A}_i (x_i - x_{i-1})$. ||

For specified G, the distribution of $G\left(\sum_1^n a_i Y_i\right)$ may be determined. Theorem 4.2 may then be used to obtain a conservative lower tolerance limit for distributions F for which $G^{-1}F$ is convex [Barlow and Proschan (1966)].

To obtain a reversal of inequality (4.2) we need Theorem 4.4 below. To prove Theorem 4.4 we state the following result:

Lemma 4.3:

$$\phi\left(\sum_1^n a_i x_i\right) - \phi(0) \geq \sum_1^n a_i\left[\phi(x_i)-\phi(0)\right] \qquad (4.4)$$

for all $0 \leq x_1 \leq\cdots\leq x_n$ and for all convex ϕ on $(-\infty,\infty)$ if and only if

$$\bar{A}_1 \geq 1,\ \ \bar{A}_2 \geq 1,\ldots,\bar{A}_k \geq 1,\ \ \bar{A}_{k+1} \leq 0,\ldots,\bar{A}_n \leq 0$$

for some k $(0 \leq k \leq n)$.

Proof:

Sufficiency. We shall prove the result for convex ϕ satisfying $\phi(0) = 0$. The more general result then follows immediately.

First suppose $\sum_1^n a_i x_i \leq x_k$. Then

$$\sum_1^k (\bar{A}_i-1)(x_i-x_{i-1}) \frac{\phi(x_i)-\phi(x_{i-1})}{x_i-x_{i-1}} + (x_k - \sum_1^n a_i x_i) \frac{\phi(x_k)-\phi(\Sigma a_i x_i)}{x_k-\Sigma a_i x_i}$$
$$\leq \sum_{k+1}^n (-\bar{A}_i)(x_i-x_{i-1}) \frac{\phi(x_i)-\phi(x_{i-1})}{x_i-x_{i-1}},$$

since (a) each ratio on the left is less than *every* ratio on the right by convexity, and (b) the sum of the coefficients on the left, $\sum_1^k (\bar{A}_i-1)(x_i-x_{i-1}) + (x_k - \sum_1^n a_i x_i)$, equals the sum of the coefficients on the right, $\sum_{k+1}^n (-\bar{A}_i)(x_i-x_{i-1})$, and (c) every coefficient is positive. After simplification, the inequality reduces to the desired result.

Next suppose $\sum_1^n a_i x_i > x_k$. Then rewrite the inequality above as

$$\sum_1^k (\bar{A}_i - 1)(x_i - x_{i-1}) \frac{\phi(x_i)-\phi(x_{i-1})}{x_i - x_{i-1}}$$

$$\leq \frac{\phi(\Sigma a_i x_i)-\phi(x_k)}{\Sigma a_i x_i - x_k} (\Sigma a_i x_i - x_k) + \sum_{k+1}^n (-\bar{A}_i)(x_i - x_{i-1}) \frac{\phi(x_i)-\phi(x_{i-1})}{x_i - x_{i-1}} .$$

The desired result follows by the same arguments as before.

Necessity. Next assume (4.4) holds for all convex ϕ. Choose $\phi(x) = x^2$, $0 = x_1 = \cdots = x_{j-1}$; $x_j = \cdots = x_n = 1$. Then (4.5) implies $\left(\sum_j^n a_i\right)^2 \geq \sum_j^n a_i$. Thus $\bar{A}_j$ is either ≥ 1 or ≤ 0.

Now we show that $\bar{A}_i \leq 0$ implies $\bar{A}_{i+1} \leq 0$. Choose $0 = x_1 = \cdots = x_{i-1} < x_i = z < x_{i+1} = \cdots = x_n$, and

$$\phi(x) = \begin{cases} 0 & \text{for } x < z \\ x - z & \text{for } x \geq z. \end{cases}$$

Choose $x_{i+1} - x_i > 0$ sufficiently small so that $\sum_1^n a_i x_i = \bar{A}_i x_i + \bar{A}_{i+1}(x_{i+1} - x_i) \leq 0$. Thus $\phi(\Sigma a_i x_i) = 0 \geq \bar{A}_{i+1}(x_{i+1} - x_i)$, by hypothesis. Hence $\bar{A}_{i+1} \leq 0$.

Finally, assume $\bar{A}_{i+1} \geq 1$. Then $\bar{A}_i$ cannot be ≤ 0 by the result just obtained. Therefore, $\bar{A}_i \geq 1$. The proof of necessity is now complete.||

Using Lemma 4.3 we may now prove

THEOREM 4.4: *Let* $G^{-1}F$ *be convex on the support of* F, $F(0) = 0 = G(0)$, *and for some* k $(0 \leq k \leq n)$, $\bar{A}_i \geq 1$, $i=1,\ldots,k$, *while* $\bar{A}_i \leq 0$, $i=k+1,\ldots,n$. *Then*

$$F\left(\sum_1^n a_i X_i\right) \overset{st}{\geq} G\left(\sum_1^n a_i Y_i\right), \tag{4.5}$$

or equivalently,

$$F\left[\sum_1^n \bar{A}_i (X_i - X_{i-1})\right] \overset{st}{\geq} G\left[\sum_1^n \bar{A}_i (Y_i - Y_{i-1})\right]. \tag{4.6}$$

Proof: Theorem 4.4 follows from Theorem 4.3 in the same way that Theorem 4.2 follows from Theorem 4.1.||

Next we obtain a comparison involving expected values of the order statistics rather than a stochastic comparison of the order statistics themselves.

THEOREM 4.5: *Let* $G^{-1}F$ *be convex on the support of* F, $F(0) = 0 = G(0)$, $a_i \geq 0$ *for* $i=1,\ldots,n$, *and* $\sum_1^n a_i \leq 1$. *Then*

$$F\left(\sum_1^n a_i EX_i\right) \leq G\left(\sum_1^n a_i EY_i\right). \tag{4.7}$$

Proof: First using Theorem 4.1 and then Jensen's inequality, we have

$$G^{-1}F\left(\sum_1^n a_i EX_i\right) \leq \sum_1^n a_i G^{-1}F(EX_i) \leq \sum_1^n a_i EG^{-1}F(X_i).$$

Since $Y_i \overset{st}{=} G^{-1}F(X_i)$, using Lemma 1 of Lehman (1959), p. 73, we obtain (4.7).||

This result was noted by Van Zwet (1964) for the case $a_i = 1$ and $a_j = 0$ for $j \neq i$, without the requirement that $F(0) = G(0) = 0$. We use (4.7) in Section 6 to obtain bounds on $\sum_1^n a_i EX_i$.

As another application of Theorem 4.2, we obtain the following inequality on weighted sums of spacings.

THEOREM 4.6: *Let $G^{-1}F$ be convex on the support of F, $F(0) = 0 = G(0)$, $\theta = \int_0^\infty x dF(x) = \int_0^\infty x dG(x)$, and $\bar{A}_i \geq 1$ for $i=1,\ldots,r$. Then*

$$P_F\left\{\sum_1^r \bar{A}_i(X_i-X_{i-1}) \geq x\right\} \geq P_G\left\{\sum_1^r \bar{A}_i(Y_i-Y_{i-1}) \geq x\right\}$$

for $x \leq \theta \min(\bar{A}_1,\ldots,\bar{A}_r)$.

<u>Proof</u>: For $c \geq \frac{1}{\min(\bar{A}_1,\ldots,\bar{A}_r)}$, by Theorem 4.5

$$F\left[\sum_1^r c\bar{A}_i(X_i-X_{i-1})\right] \overset{st}{\geq} G\left[\sum_1^r c\bar{A}_i(Y_i-Y_{i-1})\right].$$

It follows that

$$P_F\left\{F\left[\sum_1^r c\bar{A}_i(X_i-X_{i-1})\right] \geq F(\theta)\right\} \geq P_G\left\{G\left[\sum_1^r c\bar{A}_i(Y_i-Y_{i-1})\right] \geq F(\theta)\right\}.$$

By Theorem 7.1 of Barlow and Marshall (1964), p. 1256, $F(\theta) \leq G(\theta)$. It follows that

$$P_F\left\{F\left[\sum_1^r c\bar{A}_i(X_i-X_{i-1})\right] \geq F(\theta)\right\} \geq$$

$$P_G\left\{G\left[\sum_1^r c\bar{A}_i(Y_i-Y_{i-1})\right] \geq G(\theta)\right\},$$

implying

$$P_F\left\{\sum_1^r \bar{A}_i(X_i-X_{i-1}) \geq \theta/c\right\} \geq P_G\left\{\sum_1^r \bar{A}_i(Y_i-Y_{i-1}) \geq \theta/c\right\}.$$

Setting $x = \theta/c \leq \theta \min(\bar{A}_1,\ldots,\bar{A}_n)$, we obtain the desired conclusion.||

5. INEQUALITIES WHEN ONE DISTRIBUTION IS THE EXPONENTIAL

We now specialize to the case $G(x) = 1 - e^{-x}$ for $x \geq 0$. The following results are motivated by the observation that in this case the normalized spacings $(n-i+1)(Y_{in}-Y_{i-1,n})$ are independent and identically distributed for $i=1,2,\ldots,n$ and $n \geq 1$. Thus we might expect that the spacings $(n-i+1)(X_{in}-X_{i-1,n})$ would exhibit certain monotonicity properties for distributions F such that $G^{-1}F$ is convex where finite (concave on $[0,\infty)$). Such distributions F are IFR (DFR).

THEOREM 5.1: *If F is IFR (DFR) with $F(0) = 0$, then $(n-i+1)(X_{in}-X_{i-1,n})$ is stochastically increasing (decreasing) in $n \geq i$ for fixed i.*

Proof: Assume F is IFR. Let $F_{in}(x) = P[X_{in} \leq x]$ and $F_u(x) = [F(x+u) - F(u)]/\bar{F}(u)$. Then

$$P[(n-i)(X_{i+1,n}-X_{in}) > x] = \int_0^\infty \left[\bar{F}_u\left(\frac{x}{n-1}\right)\right]^{n-i} dF_{in}(u) \leq \int_0^\infty \left[\bar{F}_u\left(\frac{x}{n+1-i}\right)\right]^{n+1-i} dF_{in}(u),$$

since $[\bar{F}(t)]^{1/t}$ is decreasing in t for F IFR. Also since $\bar{F}_u(x)$ is decreasing in u for F IFR and $F_{in}(x) \leq F_{i,n+1}(x)$ for all F, we have by the lemma on p. 52 of Barlow and Proschan (1965),

$$\int_0^{x} \left[\bar{F}_u\left(\frac{x}{n+1-i}\right)\right]^{n+1-i} dF_{in}(u) \le \int_0^{\infty} \left[\bar{F}_u\left(\frac{x}{n+1-i}\right)\right]^{n+1-i} dF_{i,n+1}(u)$$

$$= P[(n+1-i)(X_{i+1,n+1}-X_{i,n+1}) > x].$$

All inequalities are reversed when F is DFR.||

Corollary 5.2: If F is IFR (DFR) and $F(0) = 0$, then $(n-i+1)(X_{in}-X_{i-1,n})$ is stochastically decreasing (increasing) in $i=1,2,\ldots,n$ for fixed n.

Proof: Assume F is IFR. First we shall show that $(n-1)(X_{2n}-X_{1n}) \overset{st}{\le} nX_{1n}$. Given X_{1n}, $X_{2n}-X_{1n}$ is the minimum of $n-1$ random variables each stochastically less than X_{1n}. Hence $X_{2n}-X_{1n} \overset{st}{\le} X_{1,n-1}$. By Theorem 5.1, $(n-1)X_{1,n-1} \overset{st}{\le} nX_{1n}$, so that $(n-1)(X_{2n}-X_{1n}) \overset{st}{\le} nX_{1n}$. The result follows by repeated conditioning.

An analogous argument applies in the DFR case.||

Next we obtain results concerning "total time on test" when successive observations are taken from an IFRA (DFRA) distribution. For example, if n items are put on life test and experimentation is terminated at the time of the r^{th} failure (censored sampling), then $T_{rn} = \sum_1^r (n-i+1)(X_{in}-X_{i-1,n})$ denotes the total time on test. This statistic has been extensively studied and applied in the case of the exponential distribution by Epstein and Sobel (1953) and Epstein (1960 a,b). The best estimate for the mean θ in the exponential case is $\hat{\theta}_{rn} = T_{rn}/r$.

THEOREM 5.3: *Let F be IFRA (DFRA), $F(0) = 0$, and $\int_0^\infty x dF(x) = \theta$. Then*

$$\text{(i)} \quad \frac{\hat{\theta}_{r,n}(\underline{X})}{\overline{X}} \equiv \frac{\sum_1^r (n-i+1)(X_i - X_{i-1})}{r\overline{X}} \overset{st}{\geq} \left(\overset{st}{\leq}\right)$$

$$\frac{\sum_1^r (n-i+1)(Y_i - Y_{i-1})}{r\overline{Y}} \equiv \frac{\hat{\theta}_{r,n}(\underline{Y})}{\overline{Y}};$$

(ii) $E\hat{\theta}_{r,n}$ *is decreasing (increasing) in* r *so that*

$E\hat{\theta}_{r,n} \geq (\leq)\ \theta$;

(iii) $\sum_1^n a_i(n-i+1)E(X_i - X_{i-1}) \leq (\geq)\ \theta \sum_1^n a_i$ *if* $a_1 \geq \cdots \geq a_n$.

Proof: (i) follows from Theorem 3.12 (iii). (ii) follows from Theorem 3.9 (i). (iii) follows from Theorem 3.9 (iv).||

Note that when F is IFR we can assert

$$H(nEX_1, (n-1)E(X_2 - X_1), \ldots, E(X_n - X_{n-1})) \geq H(\theta, \theta, \ldots, \theta)$$

when H is a Schur function.

THEOREM 5.4: *Let F be IFR (DFR) and $F(0^-) = 0$. Then*

$T_{rn} = \sum_1^r (n-i+1)(X_{in} - X_{i-1,n})$ *is stochastically increasing (decreasing) in* $n \geq r$.

Proof: Assume F is IFR. The proof is by induction on r. By Theorem 5.1 the result is true for $r=1$.

Now assume the theorem is true for $r-1$. Note that

$$P_F[T_{rn} > x] = \int_0^\infty P_{F_{u/n}}[T_{r-1,n-1} > x - u]d_u P[nX_{1n} \leq u]$$

$$\leq \int_0^\infty P_{F_{u/n}}[T_{r-1,n} > x - u]d_u P[nX_{1n} \leq u]$$

by the induction assumption since $F_{u/n}$ is IFR.

Next note that if $X_1 \leq \cdots \leq X_n$ are order statistics from any distribution F, then $P[X_2 + \cdots + X_r + (n-r)X_r > x | X_1 = w]$ is increasing in w. This is a consequence of the following two facts:

(i) Given $X_1 = w$, $X_2, \ldots, X_n$ are order statistics of a sample of size $n-1$ from the conditional distribution $P[X \leq x | X > w]$, where X has distribution F.

(ii) $P[X > x | X > w]$ is increasing in w.

It follows that $P_{F_{u/n}}[T_{r-1,n} > x - u]$ is increasing in u for any distribution F. Hence

$$\int_0^\infty P_{F_{u/n}}[T_{r-1,n} > x - u]d_u P[nX_{1n} \leq u]$$

$$\leq \int_0^\infty P_{F_{u/n}}[T_{r-1,n} > x - u]d_u P[(n+1)X_{1,n+1} \leq u]$$

$$\leq \int_0^\infty P_{F_{u/(n+1)}}[T_{r-1,n} > x - u]d_u P[(n+1)X_{1,n+1} \leq u]$$

$$= P[T_{r,n+1} > x].$$

The last inequality follows from the fact that if F is IFR, then $F_w(x)$ is decreasing in w.

A similar proof holds if F is DFR.||

Another result concerning total time on test in the case of censored sampling from an IFR distribution may be obtained directly from Theorem 4.6. Simply choose $\bar{A}_i = n-i+1,\ i=1,\ldots,r,$ in that theorem. We immediately obtain:

Corollary 5.5: Let F be IFR with mean θ and $G(t) = 1 - e^{-t/\theta}$. Then

$$P_F\left[\sum_1^r (n-i+1)(X_i-X_{i-1}) > x\right] \geq P_G\left[\sum_1^r (n-i+1)(Y_i-Y_{i-1}) > x\right]$$

for $x \leq (n-r+1)\theta$.

Next we consider truncated sampling. If n items are placed on test and successive failure times are observed until a pre-assigned time t_0, the associated sample is called a truncated sample. Let

$$V(t_0) = \sum_{i=1}^r X_i + (n-r)t_0,$$

where r denotes the number of observations $\leq t_0$, and is a random variable. $V(t_0)$ represents the total time on test up to time t_0. This statistic occurs in life testing in the exponential case. See, for example, Epstein and Sobel (1955).

THEOREM 5.6: *Let $G^{-1}F$ be starshaped on the support of F, $F(0) = 0 = G(0)$, and $\int_0^\infty x dF(x) = \int_0^\infty x dG(x)$. Then*

$$E\left[\sum_1^r X_i + (n-r)t_0\right] \geq E\left[\sum_1^s Y_i + (n-s)t_0\right], \tag{5.4}$$

where $r(s)$ *denotes the number of* $X(Y)$ *observations* $\leq t_0$.

<u>Proof</u>: Since F and G have the same mean, they cross at least once. Since $G^{-1}F$ is starshaped, F crosses G exactly once and from below. Hence there exists a least value x_0 such that $x \geq G^{-1}F(x)$ for $x \leq x_0$ while $x \leq G^{-1}F(x)$ for $x > x_0$.

Let $Y_i' = G^{-1}F(X_i)$ and let s' denote the number of $Y_1',\ldots,Y_n' \leq t_0$. Then $Y_1',\ldots,Y_n'$ and s' have the same joint distribution as $Y_1,\ldots,Y_n$ and s.

(i) Suppose $t_0 \leq x_0$. Then

$$\sum_1^r X_i + (n-r)t_0 \geq \sum_1^r Y_i' + (n-r)t_0 \geq \sum_1^{s'} Y_i' + (n-s')t_0.$$

This implies (5.4).

(ii) Suppose $t_0 > x_0$. Let

$$Y_i^* = \begin{cases} Y_i' & \text{if } Y_i' \leq t_0 \\ t_0 & \text{otherwise.} \end{cases}$$

Write

$$\sum_1^r X_i + (n-r)t_0 - \sum_1^{s'} Y_i' - (n-s')t_0$$

$$= \sum_1^r X_i + (n-r)t_0 - \sum_1^r Y_i^* - (n-r)t_0$$

$$\geq \sum_1^r (X_i - Y_i') + \sum_{r+1}^n (X_i - Y_i'),$$

since $X_i \leq Y_i'$ for $i > r$. Hence

$$E\left[\sum_1^r X_i + (n-r)t_0\right] - E\left[\sum_1^s Y_i + (n-s)t_0\right]$$

$$\geq E\sum_1^n X_i - E\sum_1^n Y_i = 0. ||$$

Remark: In the special case in which G is the exponential distribution and F is an IFRA distribution, then $G^{-1}F$ is starshaped. In this case, (5.4) yields a lower bound on the expected total time on test in truncated sampling from an IFRA distribution with known mean.

6. BOUNDS ON EXPECTED VALUES OF ORDER STATISTICS FROM MONOTONE FAILURE RATE DISTRIBUTIONS

In Section 3 we obtained explicit bounds on $EX_{i,n}$ assuming $G^{-1}F$ is starshaped [cf. (3.7)]. In particular, if F is IFRA with mean θ, we have the result

$$\frac{\theta \sum_{j=1}^{i} 1/(n-j+1)}{\sum_{j=1}^{i} 1/j} \leq EX_{i,n} \leq n\theta \sum_{j=1}^{i} 1/(n-j+1)$$

for $1 \leq i < n$ and

$$\theta \leq EX_{n,n} \leq \theta \sum_{j=1}^{n} Y_j,$$

for $i=n$. The bounds are non-trivial but only sharp for $i=1$ or $i=n$.

If F is DFRA with mean θ we have, using (3.6),

$$0 \le EX_{1,n} \le \theta/n,$$

$$0 \le EX_{i,n} \le \frac{i\theta \sum_{1}^{i} 1/(n-j+1)}{\sum_{1}^{i} 1/j} \quad \text{for} \quad 1 < i < n,$$

$$\theta \sum_{1}^{n} \frac{1}{j} \le EX_{n,n} \le n\theta.$$

All lower bounds are sharp. To see this, let

$$\bar{F}(x) = \begin{cases} 0 & x < 0 \\ \varepsilon e^{-\frac{\varepsilon x}{\theta}} & x \ge 0, \end{cases}$$

where $0 < \varepsilon \le 1$. Then F is DFR with mean θ, and for $1 \le i < n$,

$$P[X_i \ge x] = \sum_{j=0}^{i-1} \binom{n}{j} [F(x)]^j [\bar{F}(x)]^{n-j}$$

$$\le \sum_{j=0}^{i-1} \binom{n}{j} \varepsilon^{n-j} e^{-\frac{\varepsilon x(n-j)}{\theta}}.$$

Hence $EX_i = \int_0^\infty P[X_i \ge x]dx < 2^n \varepsilon\theta$. Since we can choose ε arbitrarily close to 0, we see that

$$EX_i \ge 0 \quad (1 \le i < n)$$

is sharp. Note that since $\sum_{1}^{n} EX_{i,n} = n\theta$, $EX_{n,n}$ approaches $n\theta$ as ε decreases to 0. Hence the upper bound for $i=n$ is also sharp. The upper

bound for $i=1$ is attained by the exponential. The other bounds are non-trivial but not sharp.

Using Theorem 4.5 we can obtain additional explicit upper bounds on $\sum_1^n a_i EX_i$ assuming F is IFR, $a_i \geq 0$ and $\sum_1^n a_i \leq 1$.

THEOREM 6.1: *If F is IFR with mean θ, $F(0) = 0$, $a_i \geq 0$ for $i=1,\ldots,n$, and $\sum_1^n a_i \leq 1$, then*

$$\sum_1^n a_i EX_i \leq \theta \sum_1^n a_i EY_i \Big/ \left[1 - \exp\left(- \sum_1^n a_i EY_i\right)\right] \tag{6.1}$$

where $EY_i = \sum_{j=1}^{i} 1/(n-j+1)$.

<u>Proof</u>: We may assume without loss of generality that $\theta = 1$. As shown in Barlow and Marshall (1964),

$$F(x;1) \geq b(x;1) = \begin{cases} 0 & x \leq 1 \\ 1 - e^{-wx} & x \leq 1, \end{cases}$$

where w depends on x and satisfies

$$1 - e^{-wt} = w. \tag{6.2}$$

Since

$$F\left[\sum_1^n a_i EX_i\right] \leq 1 - \exp\left[- \sum_1^n a_i EY_i\right]$$

by Theorem 4.5, we have

$$b\left[\sum_1^n a_i EX_i ; 1\right] \le 1 - \exp\left[-\sum_1^n a_i EY_i\right].$$

Choose t such that

$$b(t;1) = 1 - e^{-wt} = 1 - \exp\left[-\sum_1^n a_i EY_i\right], \tag{6.3}$$

where w depends on t. It follows that

$$\sum_1^n a_i EX_i \le t = \sum_1^n a_i EY_i / w.$$

Using (6.2) and (6.3) we obtain (6.1).||

Sharp bounds on expected values of order statistics from an IFR distribution can be given in terms of the p^{th} percentile.

THEOREM 6.2: *Let F be IFR with p^{th} percentile ξ_p. Then*

$$EX_j \le \max\left\{\xi_p, \frac{p}{-\log q}\left(\frac{1}{n} + \cdots + \frac{1}{n-j+1}\right)\right\} \tag{6.4}$$

and

$$EX_j \ge \sum_{i=0}^{j-1} \binom{n}{i} \int_0^{\xi_p} \left(1 - e^{x \log q/\xi_p}\right)^i \left(e^{x \log q/\xi_p}\right)^{n-i} dx \tag{6.5}$$

where $q = 1 - p$. All inequalities are sharp.

__Proof:__ To show (6.4), let

$$G_\Delta(x) = \begin{cases} 1 & \text{for } 0 \le x \le \Delta \\ q \exp\left[\dfrac{x-\xi_p}{\xi_p - \Delta} \log q\right] & \text{for } x > \Delta. \end{cases}$$

Note that $\bar{G}_\Delta(\Delta) = 1$ and $\bar{G}_\Delta(\xi_p) = q$. Since $\log \bar{F}(x)$ is concave, there exists at least one value of $\Delta \geq 0$ such that $\bar{G}_\Delta(x) \geq \bar{F}(x)$ for all $x \geq 0$. Thus $EX_j \leq \sup_{0 \leq \Delta \leq \xi_p} EY_j$ where Y_j is the j^{th} order statistic from G_Δ. Now

$$EY_j = \Delta + \int_\Delta^\infty \sum_{i=0}^{j-1} \binom{n}{i} [G_\Delta(x)]^i [\bar{G}_\Delta(x)]^{n-i} dx$$

$$= \Delta + \int_\Delta^\infty \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_{G_\Delta(x)}^1 t^{j-1}(1-t)^{n-j} dt dx$$

by p. 234, Mood (1950).

To find the maximizing Δ, consider

$$\frac{\partial}{\partial \Delta} EY_j = 1 - \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_{G_\Delta(\Delta)}^1 t^{j-1}(1-t)^{n-j} dt$$

$$+ \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_\Delta^\infty [G_\Delta(x)]^{j-1}[\bar{G}_\Delta(x)]^{n-j} q \exp\left[\frac{x-\xi_p}{\xi_p-\Delta} \log q\right] \log q \frac{x-\xi_p}{(\xi_p-\Delta)^2} dx.$$

Since $G_\Delta(\Delta) = 0$, $-(\xi_p - \Delta) \frac{\partial EY_j}{\partial \Delta}$ reduces to

$$\frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} \int_\Delta^\infty [G_\Delta(x)]^{j-1}[\bar{G}_\Delta(x)]^{n-j} g_\Delta(x)(x-\xi_p) dx = EY_j - \xi_p$$

where g_Δ is the density of G_Δ. Hence

$$-(\xi_p-\Delta) \frac{\partial EY_j}{\partial \Delta} = \Delta - \frac{\xi_p-\Delta}{\log q} \left(\frac{1}{n} + \cdots + \frac{1}{n-j+1}\right) - \xi_p$$

$$= -(\xi_p-\Delta)\left[1 + \frac{1}{\log q}\left(\frac{1}{n} + \cdots + \frac{1}{n-j+1}\right)\right].$$

For j such that $1 + \frac{1}{\log q}\left(\frac{1}{n} + \cdots + \frac{1}{n-j+1}\right) \leq 0$, we have $\frac{\partial EY_j}{\partial \Delta} \leq 0$.

For j such that $1 + \frac{1}{\log q}\left(\frac{1}{n} + \cdots + \frac{1}{n-j+1}\right) \geq 0$, we have $\frac{\partial EY_j}{\partial \Delta} \geq 0$.

Thus EY_j is maximized in the first case at $\Delta = 0$ and in the second case at $\Delta = \xi_p$. When $\Delta = 0$, $EY_j = \frac{\xi_p}{-\log q}\left(\frac{1}{n} + \cdots + \frac{1}{n-j+1}\right)$; when $\Delta = \xi_p$, $EY_j = \xi_p$.

To show (6.5), let

$$\bar{G}(x) = \begin{cases} e^{x \log q/\xi_p} & \text{for } 0 \leq x < \xi_p \\ 0 & \text{for } \xi_p \leq x < \infty. \end{cases}$$

G has p^{th} percentile ξ_p. Moreover, since $\log \bar{F}$ is concave, it cannot cross $\log \bar{G}$ on $(0,\xi_p)$; hence $\bar{G}(x) \leq \bar{F}(x)$ for all $x \geq 0$. Thus $EY_j \leq EX_j$, where Y_j is the j^{th} order statistic from G. But

$$EY_j = \int_0^\infty \sum_{i=0}^{n-1} \binom{n}{i} [G(x)]^i [\bar{G}(x)]^{n-i} dx.$$

Eq. (6.5) follows from the definition of G given just above.||

7. PROPERTIES PRESERVED IN TAKING ORDER STATISTICS FROM IFR (DFR) DISTRIBUTIONS

In Barlow and Proschan (1965), pp. 38-39, it is shown that order statistics from an IFR distribution themselves have an IFR distribution. This is not true for spacings from an IFR distribution. To see this, suppose that F is IFR with mass at $a < \infty$. Then the distribution of $X_2 - X_1$ will have a jump at the origin, and hence cannot be IFR. The reverse situation exists for DFR distributions. Order statistics from DFR distributions are not necessarily DFR.

This is evident since the exponential is DFR, while the i^{th} order statistic from the exponential is strictly IFR for $i > 1$. However, spacings from a DFR distribution are DFR.

THEOREM 7.1: *If F is DFR, then $X_i - X_{i-1}$ has a DFR distribution, $i=1,2,\ldots,n$.*

Proof: Let H_i denote the distribution of $X_i - X_{i-1}$. For $i=1$, $\bar{H}_1(x) = [\bar{F}(x)]^n$, so that H_1 is DFR. For $i=2,\ldots,n$, write

$$\bar{H}_i(x) = \frac{n!}{(i-2)!(n-i+1)!}\int_0^\infty [F(u)]^{i-2}[\bar{F}(u+x)]^{n-i+1}dF(u) \quad \text{for} \quad i=2,\ldots,n.||$$

Now $\bar{F}(u+x)$ is logarithmically convex in $x \geq 0$ since F is DFR. Hence, so is $[\bar{F}(u+x)]^{n-i+1}$ and therefore $\bar{H}_i(x)$ is, since it is a positively weighted linear combination of logarithmically convex functions [Artin (1931)]. Thus $H_i(x)$ is DFR for fixed $i=2,\ldots,n.||$

A stronger property than IFR is the property that F has density f such that $\log f(x)$ is concave where finite; i.e., f is PF_2. Order statistics do preserve the PF_2 property, as shown in

THEOREM 7.2: *Suppose f is PF_2, with $f(x)$ not necessarily 0 for negative x. Then the density f_{in} of the i^{th} order statistic is also PF_2 for fixed $i=1,2,\ldots,n$.*

Proof: When f is PF_2, so is F and $\bar{F}$. Thus

$$f_i(x) = \frac{n!}{(i-1)!(n-i)!}F^{i-1}(x)\bar{F}^{n-i}(x)f(x)$$

is also logarithmically concave. Equivalently, f is $PF_2.||$

THEOREM 7.3: *Let f be PF_2 with $f(x)$ not necessarily 0 for $x < 0$. Then h_i, the density of $X_i - X_{i-1}$ is also PF_2 for fixed $i=2,\ldots,n$. If $f(x) = 0$ for $x < 0$, then h_1 is PF_2, where h_1 is the density of X_1.*

Proof: Note

$$h_i(x) = \frac{n!}{(i-2)!(n-i)!} \int_{-\infty}^{\infty} F^{i-2}(u)f(u)f(u+x)\bar{F}^{n-i}(u+x)du$$

for $i=2,3,\ldots,n$. Since f is PF_2, so is $r(u) = F^{i-2}(-u)f(-u)$, $s(u) = f(u)\bar{F}^{n-1}(u)$. Since the PF_2 property is preserved under convolution,

$$h_i(x) = \frac{n!}{(i-2)!(n-i)!} \int_{-\infty}^{\infty} r(-u)s(u+x)du$$

is PF_2 for fixed $i=2,3,\ldots,n$.

Assuming $f(x) = 0$ for $x < 0$, we see that h_1 is PF_2 from

$$h_1(x) = nf(x)\bar{F}^{n-1}(x). ||$$

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